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HYDRODYNAMIC THEORY OF THE ACTION OF EXPLOSION AND THE ASSUMPTION OF INCOMPRESSIBILITY

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It is shown in this paper that in the volume-compressible medium (the motion of which is described by a scalar wave equation) the limiting mean pressure pulse is a harmonic function of position. For this reason the dynamic pressure and velocity fields computed according to the "incompressibility scheme" describe a certain integral asymptotics, for $t \to \infty$, for the corresponding compressible medium.

Experiments in [1] indicate that the motion of the continuous medium accompanying the explosion, can be separated into two stages. The first stage, of a very short duration, is characterized by expansion of the stress wave and relatively small increases in the particle displacements. During this stage reflections appear and disruption may occur. During the second stage the motion of the particles of the medium, which is already partially disrupted, is developed further. This is the ballistic stage during which ejection takes place in the nonunderground explosions, or throwing of projectiles when a directional explosion takes place. The velocity field developed by the end of the first stage, serves as the "initial" field for the ballistic stage.

It is assumed that during the first stage the motion is mainly determined by the inertial resistance of the medium, the compressibility playing a secondary role. Since the pressure developed during the initial stage of the explosion is very large, it is reasonable to make another assumption that the tensile properties of the medium are also of secondary importance and describe the state of stress by a spherical pressure tensor. Thus for the first stage we have the following set of assumptions: (a) the medium is incompressible, (b) the medium is ideal (tangential stresses are absent) and (c) strains and displacements remain small. We shall call the set of these hypotheses the "incompressibility scheme".

Using the framework of this scheme to consider some examples we find that the results of our analysis give, in a number of cases, a satisfactory agreement with experiments [2]. The nature of the incompressibility scheme can be clarified and its region of applicability indicated, by assessing its relation to the allied problems in which the wave propagation is taken into account. The present paper deals with one of the possible correlations.

1. Let an arbitrary region B whose boundary is S, be specified. The boundary may consist of one or several piecewise smooth surfaces. The region B may extend to infinity and a solution of the Dirichlet problem for the Laplace equation must exist for this region. In addition, B is filled with an ideal compressible medium in which a purely linear (under small deformations) compression law

$$\partial p \,/\, \partial t = -k^2 \operatorname{div} \mathbf{v}, \quad k^2 > 0 \tag{1.1}$$

where $p\ (M,\ t)$ is the pressure and $\mathbf{v}\ (M,\ t)$ is the velocity of a particle, holds.

Let the impulsive pressure

$$p\ (Q,\ t) = f\ (Q,\ t), \quad Q \in S \tag{1.2}$$
$$f\ (Q,\ t) \equiv 0, \quad t < 0, \quad T < t$$

be specified at the points of the boundary surface S. The initial conditions are assumed null

$$p\ (M,\ 0) = 0, \quad \mathbf{v}\ (M,\ 0) = 0, \quad M \in B + S$$

and our aim is to determine the pressure and velocity fields at $t > 0$.

Using the equation of motion (in the linear approximation)

$$\rho_0 \partial \mathbf{v} \,/\, \partial t + \operatorname{grad} p = 0 \tag{1.3}$$

and eliminating $\bar{v}\ (M,\ t)$ from (1.1) and (1.3), we obtain for $p\ (M,\ t)$ the following wave equation:

$$\Delta p - \frac{1}{a^2} \frac{\partial^2 p}{\partial t^2} = 0, \qquad a^2 = \frac{k^2}{\rho_0} \tag{1.4}$$

where ρ_0 is the density, assumed to be constant.

Thus the problem of determining the motion of the medium under the action of a specified load applied at the boundary, has been reduced to a mixed boundary value problem for a wave equation. When $p\ (M,\ t)$ has been found, the velocity field is obtained by the formula

$$\mathbf{v}\,(M, t) = -\frac{1}{\rho_0} \int_0^t \operatorname{grad} p\,(M, \tau)\, d\tau \tag{1.5}$$

The incompressibility scheme offers a simpler procedure. It considers the medium as incompressible and assumes that a characteristic time τ $(T \ll \tau)$ exists, at which the process is established. Then, for the pressure pulse $\Pi\ (M)$

$$\Pi\,(M) = \int_0^\tau p\,(M, t)\, dt$$

the condition of incompressibility implies that $\Delta \Pi = 0$. Relations (1.2) yield the boundary values for the pulse

$$\Pi\,(Q) = \int_0^\tau f\,(Q, t)\, dt$$

As a result, the incompressibility scheme reduces the problem to the Dirichlet problem for the Laplace equation. The problem proposed for the velocity field is

$$\mathbf{v} = -\rho_0^{-1} \operatorname{grad} \Pi \tag{1.6}$$

(transposition of the integral and the derivative). The functions p and $\mathbf{v}$ computed in this manner are independent of time. This agrees with the supposition that the time in which the asymptotics of the process is developed, is extraordinarily short.

2. We now introduce the pressure pulse $\Pi\ (M,\ t)$, the mean pressure pulse $P\ (M,\ t)$ and the limiting mean pressure pulse $P_0\ (M)$

$$\Pi(M,t)=\int_0^t p(M,\tau)\,d\tau$$

$$P(M,t)=\frac{1}{t}\int_0^t \Pi(M,\tau)\,d\tau \qquad (2.1)$$

$$P_0(M)=\lim_{t\to\infty}\left(\frac{1}{t}\int_0^t \Pi(M,\tau)\,d\tau\right)=\lim_{t\to\infty}\left(\frac{1}{t}\int_0^t p(M,\tau)(t-\tau)\,d\tau\right)$$

We shall also consider the mean particle velocity $\mathbf{V}(M,t)$ and the limiting mean particle velocity $\mathbf{V}_0(M)$

$$\mathbf{V}(M,t)=\frac{1}{t}\int_0^t \mathbf{v}(M,\tau)\,d\tau, \qquad \mathbf{V}_0(M)=\lim_{t\to\infty}\mathbf{V}(M,t) \qquad (2.2)$$

We assume that the above limits exist.

The principal result of this paper consists of a proof of the following assertion: if $p(M,t)$ is a solution of (1.4) with the boundary conditions (1.2) and null initial conditions satisfying the limit relation (*)

$$\lim_{t\to\infty}\frac{p(M,t)}{t}=0, \qquad M\in B$$

uniformly in B, then $P_0(M)$ is a function which is harmonic in B and assumes the following values at the boundary S :

$$P_0(Q)=F_0(Q), \quad Q\in S, \quad F_0(Q)=\lim_{t\to\infty}\left(\frac{1}{t}\int_0^t f(Q,\tau)(t-\tau)\,d\tau\right)$$

In addition we have $\mathbf{V}_0(M)=-\rho_0^{-1}\operatorname{grad}P_0(M)$.

Proof. We assume that the function $P(M,t)$ is discontinuous, since the propagation of the waves generated by a boundary (or interior point) pulse is accompanied by the propagation of the surfaces of discontinuity which are reflected from the boundary of B. Let $N(M,t)$ be the number of the surfaces of discontinuity passing during the interval of time $(0,t)$ through the point M. When the function $P(M,t)$ is differentiated with respect to the coordinates, the operation of differentiation cannot be simply inserted under the integral sign. Let us denote by $t_j(M)$ the instant at which the jth surface of discontinuity passes through the point M, and split the interval of the integration as follows:

$$P(M,t)=\frac{1}{t}\int_0^{t_1} p(M,\tau)(t-\tau)\,d\tau+\frac{1}{t}\sum_{j=1}^{N}\int_{t_j}^{t_{j+1}} p(M,\tau)(t-\tau)\,d\tau+$$
$$+\frac{1}{t}\int_{t_N}^{t} p(M,\tau)(t-\tau)\,d\tau$$

*) It is very probable that this condition is superfluous and arises naturally from the finite character and the boundedness of the boundary function. In the present paper however, this aspect of the problem is not considered.

Taking into account the dependence of the limits of integration on the coordinates, we find the Laplace operator of $P(M, t)$ using the formula $\Delta = \operatorname{div}(\operatorname{grad})$. We obtain

$$\Delta P(M, t) = \sum_{j=1}^{N} \left(1 - \frac{t_j}{t}\right)\left\{[p]_j \Delta t_j + \left[\frac{\partial p}{\partial t}\right]_j \operatorname{grad}^2 t_j + 2(\operatorname{grad} p_j \cdot \operatorname{grad} t_j)\right\} -$$

$$- \frac{1}{t}\sum_{j=1}^{N} [p]_j \operatorname{grad}^2 t_j + \frac{1}{2}\int_0^{t_1} \Delta p(t-\tau)\, d\tau + \frac{1}{t}\sum_{j=1}^{N}\int_{t_j}^{t_{j+1}} \Delta p(t-\tau)\, d\tau +$$

$$+ \frac{1}{t}\int_{t_N}^{t} \Delta p(t-\tau)\, d\tau$$

The brackets denote the jumps in the value of the function at the surface of discontinuity, i.e. $[p]_j = p_j^- - p_j^+$, etc. Simplifications which can be introduced in this formula ensue from the following. The function $p(M, t)$ is a solution of (1.4), therefore

$$\int_{t_j}^{t_{j+1}} \Delta p(t-\tau)\, d\tau = \frac{1}{a^2}\int_{t_j}^{t_{j+1}} \frac{\partial^2 p}{\partial \tau^2}(t-\tau)\, d\tau = \frac{1}{a^2}\left[\frac{\partial p}{\partial t}(t-\tau)\right]_{t_j}^{t_{j+1}} + \frac{1}{a^2}[p]_{t_j}^{t_{j+1}}$$

The surface of discontinuity in the (x, y, z, t)-space is a characteristic surface, the equation of which is $t = t_j(M)$, therefore $\operatorname{grad} t_j = \mathbf{n}_j / a$ where $\mathbf{n}_j$ is a vector of the normal to the surface of discontinuity at the point M. From this follow

$$(\operatorname{grad} p \cdot \operatorname{grad} t_j) = \frac{1}{a}\frac{\partial p}{\partial n} = \frac{1}{a}\frac{\partial p}{\partial s}$$

provided that s is the arc length counted along the ray (the ray is considered in the coordinate space). Collecting the like terms we obtain

$$\Delta P(M, t) = \sum_{j=1}^{N}\left(1 - \frac{t_j}{t}\right)\left\{[p]_j \Delta t_j + \frac{2}{a}\left(\left[\frac{\partial p}{\partial s}\right]_j + \frac{1}{a}\left[\frac{\partial p}{\partial t}\right]_j\right)\right\} + \frac{p(M, t)}{at} \quad (2.3)$$

All terms in (2.3) except the last one are equal to zero. This follows from the conditions at the fronts at which the solutions of the wave equation become discontinuous, and which are known. The general theory of hyperbolic equations can be found in e.g. [3], ch. 4. We give a brief version of the derivation leading directly to the result in the required form. We introduce a radial, orthogonal (α, β, s)-coordinate system attached to some surface of discontinuity, where s denotes the arc length counted along the ray, while α and β denote the curvilinear coordinates at the surface of discontinuity. The coordinate axes $\alpha = \text{const}$ and $\beta = \text{const}$ are the lines of curvature on this surface.

We introduce the following system of functions:

$$\eta_k(\tau) = \begin{cases} 0, & \tau > 0 \\ \tau^k, & \tau \leqslant 0 \end{cases} \qquad (k = 1, 2, 3, \ldots)$$

We write $p(M, t)$ near the surface of discontinuity in the form of an expansion

$$p(M, t) = p_0^- \eta_0(s - s_1) + p_1^- \eta_1(s - s_1) + \ldots + p_0^+ \eta_0(s_1 - s) + p_1^+ \eta_1(s_1 - s) + \ldots \quad (2.4)$$

Here the plus and minus superscripts denote the limiting values at each side of the surface, the coefficients p_k^- and p_k^+ are functions of α, β and $s_1 \equiv at$. The requirement that $p(M, t)$ be a solution of (1.4) imposes certain conditions on the coefficients p_k^-

and p_k^+. These coefficients can be obtained by "substituting" the expansion (2.4) into Eq. (1.4). Let us denote the set of terms of the expansion, the leading term of which contains η_k, by $E(\eta_k)$, so that e. g.

$$E(\eta_k) = a_k\eta_k + a_{k+1}\eta_{k+1} + a_{k+2}\eta_{k+2} + \dots$$

Inserting this into Eq. (1.4) we obtain

$$\Delta p - \frac{1}{a^2}\frac{\partial^2 p}{\partial t^2} = \operatorname{div}\operatorname{grad} p - \frac{1}{a^2}\frac{\partial^2 p}{\partial t^2} =$$

$$= \left\{\frac{1}{H_\alpha H_\beta}\frac{\partial (H_\alpha H_\beta)}{\partial s}(p_0^- - p_0^+) + 2\left(\frac{\partial p_0^-}{\partial s_1} - \frac{\partial p_0^+}{\partial s_1}\right)\right\}\delta(s - s_1) + E(\eta_0) = 0$$

From this follows the necessary condition

$$\frac{1}{H_\alpha H_\beta}\frac{\partial (H_\alpha H_\beta)}{\partial s}(p_0^- - p_0^+) + 2\left(\frac{\partial p_0^-}{\partial s_1} - \frac{\partial p_0^+}{\partial s_1}\right) = 0$$

Alternatively, taking into account the fact that

$$p_0^- - p_0^+ = [p], \quad \frac{\partial p_0^-}{\partial s_1} - \frac{\partial p_0^+}{\partial s_1} = \left[\frac{\partial p}{\partial s}\right] + \frac{1}{a}\left[\frac{\partial p}{\partial t}\right]$$

$$\operatorname{grad} t_j = \frac{\mathbf{n}_j}{a}, \qquad \Delta t_j = \operatorname{div}(\operatorname{grad} t_j) = \frac{1}{a}\frac{1}{H_\alpha H_\beta}\frac{\partial (H_\alpha H_\beta)}{\partial s}$$

we obtain

$$[p]\,\Delta t_j + \frac{2}{a}\left(\left[\frac{\partial p}{\partial s}\right] + \frac{1}{a}\left[\frac{\partial p}{\partial t}\right]\right) = 0$$

Thus (2.3) leads to the equation

$$\Delta P(M, t) = p(M, t) / at \tag{2.5}$$

Let $G(M, M_1)$ be the Green's function for the region B. Then from (2.5) follows

$$P(M, t) = \int_B \frac{1}{at}\, p(M_1, t)\, G(M, M_1)\, dx_{M_1} + H(M, t) \tag{2.6}$$

where $H(M, t)$ is a function which is harmonic in B and satisfies the following boundary condition on S:

$$H|_S = \frac{1}{t}\int_0^t f(Q, \tau)(t - \tau)\, d\tau$$

If $f(Q, t)$ is finite (as it was assumed) and uniformly bounded, then the boundary values converge uniformly as $t \to \infty$. This follows from the estimate ($T < t, t_1$)

$$\left|\frac{1}{t}\int_0^T f(Q, \tau)(t - \tau)\, d\tau - \frac{1}{t_1}\int_0^T f(Q, \tau)(t_1 - \tau)\, d\tau\right| < \left(\frac{1}{t} + \frac{1}{t_1}\right)\text{const}$$

By the Weierstrass theorem, the harmonic function $H(M, t)$ converges, as $t \to \infty$, uniformly in $B + S$ to the limit $H_0(M)$ which itself is also a harmonic function. The integral in (2.6) tends to zero. Consequently

$$\lim_{t\to\infty} P(M, t) = P_0(M) = H_0(M)$$

and $P_0(M)$ is therefore a harmonic function satisfying the boundary condition $P_0|_S = F_0(Q)$, QED (*).

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*) Here the author expresses his gratitude to G. I. Eskin, who suggested that the proof could be simplified by introducing the concept of a generalized solution of a differential equation.

Now we shall show the relation connecting the limiting mean velocity $\mathbf{V}_0(M)$ with the limiting pulse $P_0(M)$. Termwise integration of (1.3) yields

$$\int_0^t \frac{\partial \mathbf{v}}{\partial \tau} d\tau = \int_0^{t_1} \frac{\partial \mathbf{v}}{\partial \tau} d\tau + \sum_{j=1}^{N} \int_{t_j}^{t_{j+1}} \frac{\partial \mathbf{v}}{\partial \tau} d\tau + \int_{t_N}^{t} \frac{\partial \mathbf{v}}{\partial \tau} d\tau =$$

$$= \mathbf{v}(M, t) + \sum_{j=1}^{N} [\mathbf{v}]_j = \frac{1}{\rho_0} \operatorname{grad} \int_0^t p(M, \tau)\, d\tau + \frac{1}{\rho_0} \operatorname{grad} t_j \sum_{j=1}^{N} [p]$$

Since $[\mathbf{v}] = (\mathbf{v}_j^- - \mathbf{v}_j^+) \uparrow\uparrow \mathbf{n}_j$ and $\operatorname{grad} t_j = \mathbf{n}_j / a$, we obtain, taking also into account the condition at the discontinuity $[p]_j = \rho_0 a [v]_j$,

$$\mathbf{v}(M, t) = -\frac{1}{\rho_0} \operatorname{grad} \int_0^t p(M, \tau)\, d\tau \qquad (2.7)$$

Thus we found it possible to perform a simple interchange of the grad operator and the integral in spite of the presence of the surfaces of discontinuity. The integral in (2.7) is continuous, hence

$$\mathbf{V}(M, t) = -\frac{1}{\rho_0} \operatorname{grad} \left(\frac{1}{t} \int_0^t d\tau \int_0^{\tau} p(M, \tau_1)\, d\tau_1 \right)$$

Passing to the limit as $t \to +\infty$, we obtain

$$\mathbf{V}_0(M) = -\rho_0^{-1} \operatorname{grad} P_\theta(M)$$

3. What we have said, implies, that we can interpret the incompressibility scheme as an integral asymptotics at $t \to \infty$ for the corresponding problem on a compressible medium. However, the incompressibility scheme can be used successfully as an approximate method only if the problem to be solved depends on the integral effect of the explosion and, if the velocity of wave propagation is large. In other words, it is required that the time in which a wave covers the characteristic distance, is of the order of the time of action of the source. The observer perceives the final result without going into the details of the process.

Carelessness in application of the scheme may lead to erroneous conclusions. In particular, the kinetic energy calculated from the limiting velocity $\mathbf{V}_0$ may differ from the actual limiting mean energy. This problem however requires a separate investigation.

An analogous asymptotic approach to the problems of dynamic theory of elasticity could be of interest. Such an investigation would lead to formulating a static problem describing an integral asymptotics.

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